

ON COHOMOGENEITY ONE BIHARMONIC HYPERSURFACES INTO THE EUCLIDEAN SPACE

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ABSTRACT. The aim of this paper is to prove that there exists no cohomogeneity one G -invariant proper biharmonic hypersurface into the Euclidean space \mathbb{R}^n , where G denotes a transformation group which acts on \mathbb{R}^n by isometries, with codimension two principal orbits. This result may be considered in the context of the Chen conjecture, since this family of hypersurfaces includes examples with up to seven distinct principal curvatures. The paper uses the methods of equivariant differential geometry. In particular, the technique of proof provides a unified treatment for all these G -actions.

1. INTRODUCTION

According to B.-Y. Chen [6] an immersion $\varphi : M^m \hookrightarrow \mathbb{R}^n$ is called *biharmonic* if

$$\Delta H = (\Delta H_1, \dots, \Delta H_n) = 0, \quad (1.1)$$

where $H = (H_1, \dots, H_n)$ is the mean curvature vector field and Δ denotes the Beltrami-Laplace operator on M (our sign convention is such that $\Delta h = -h''$ when h is a function of one real variable). It follows from Beltrami's equation

$$m H = -(\Delta \varphi_1, \dots, \Delta \varphi_n)$$

that the biharmonicity condition is equivalent to

$$\Delta^2 \varphi = (\Delta^2 \varphi_1, \dots, \Delta^2 \varphi_n) = 0,$$

which justifies the previous definition of biharmonic immersions.

The study of biharmonic immersions in \mathbb{R}^n can be set in a more general variational, Riemannian geometric context. More precisely, we recall that a smooth map $\varphi : (M, g) \rightarrow (N, h)$ is a *harmonic map* if it is a critical point of the *energy* functional

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g, \quad (1.2)$$

whose Euler-Lagrange equation is $\tau(\varphi) = \text{trace } \nabla d\varphi = 0$. A natural generalization of harmonic maps are the so-called *biharmonic maps*: these maps are the critical

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points of the bienergy functional (as suggested by Eells–Lemaire [10])

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g . \quad (1.3)$$

In [13] G. Jiang showed that the Euler-Lagrange equation associated to $E_2(\varphi)$ is given by $\tau_2(\varphi) = 0$, where the *bitension field* $\tau_2(\varphi)$ is

$$\tau_2(\varphi) = -\Delta\tau(\varphi) - \text{trace } R^N(d\varphi, \tau(\varphi))d\varphi . \quad (1.4)$$

An immersed submanifold into a Riemannian manifold (N, h) is called a *biharmonic submanifold* if the immersion is a biharmonic map. In particular, minimal immersions are trivially biharmonic, so that we call *proper* biharmonic any biharmonic immersion which is not minimal. We observe that when $N = \mathbb{R}^n$ the curvature term in (1.4) vanishes and $\tau_2(\varphi) = 0$ is equivalent to (1.1). Thus the definition of biharmonic immersed submanifolds extends the original definition of Chen.

In the case of hypersurfaces, biharmonicity can be expressed by means of the following general result (see [2, 7, 19, 23, 24]):

Theorem 1.1. *Let $\varphi : M^{n-1} \rightarrow N^n$ be an isometric immersion with mean curvature vector field $H = (f/(n-1))\eta$. Then φ is biharmonic if and only if the normal and the tangent components of $\tau_2(\varphi)$ vanish, i.e.,*

$$\Delta f + f|A|^2 - f \text{Ricci}^N(\eta, \eta) = 0 \quad (1.5a)$$

and

$$2A(\text{grad } f) + f \text{grad } f - 2f \text{Ricci}^N(\eta)^\top = 0 \quad (1.5b)$$

respectively, where A is the shape operator and $\text{Ricci}^N(\eta)^\top$ is the tangent component of the Ricci tensor field of N in the direction of the unit normal vector field η of M in N .

We shall work in the framework of equivariant differential geometry, so let N be a Riemannian manifold and $I(N)$ its full isometry group. It is well-known (see [22]) that $I(N)$ is a Lie group which acts differentiably on N . A Lie subgroup G of $I(N)$ is called an *isometry group* of N and, following [18], we recall that its *cohomogeneity* is defined as the codimension in N of the maximal dimensional orbits, also called the *principal* orbits (of course, all the orbits are homogeneous spaces, since they are of the type G/H , where H is the stabilizer). The *cohomogeneity* of a G -invariant submanifold M of N is defined as the dimension of M minus the dimension of the principal orbits.

In this paper we shall be interested in the case that N is the Euclidean space \mathbb{R}^n and the cohomogeneity of G is two, so that G -invariant hypersurfaces are *cohomogeneity one submanifolds*. This type of isometry groups of \mathbb{R}^n have been fully classified in [18]. In particular, Hsiang and Lawson, developing the work of various famous authors, including Cartan and Weil, divided the cohomogeneity two isometry groups G acting on \mathbb{R}^n into five types according to the geometric shape of their orbit space $Q = \mathbb{R}^n/G$, which is a linear cone in \mathbb{R}^2 of angle π/d , $d = 1, 2, 3, 4, 6$ respectively. In this context, a G -invariant hypersurface in \mathbb{R}^n can be completely described by

means of its profile curve γ into the orbit space Q . In particular, it turns out that a G -invariant hypersurface is a biharmonic submanifold if and only if the curve γ satisfies a certain system of ordinary differential equations (see Proposition 2.1 in Section 2 below for details). This approach, which uses the symmetries deriving by the group action to reduce a PDE's problem to an ODE's system, has been very fruitful in various context (construction of harmonic maps, counterexamples for Bernstein's type problems for minimal and CMC immersions (see, for example, [3, 12, 17])). In general, reduction to an ODE has been a valuable tool because it has helped to produce new solutions. By contrast, in our case what we obtain is a nonexistence result in the direction of the still open Chen conjecture (see [5, 6] and [4, chapter 7]): *biharmonic submanifolds into \mathbb{R}^n are minimal*.

Chen's conjecture is still open even for biharmonic hypersurfaces in \mathbb{R}^n though, by a result of Dimitric (see [8]), we know that any biharmonic hypersurface in \mathbb{R}^n with at most two distinct principal curvatures is minimal. Other partial results for low dimensions state that biharmonic hypersurfaces with at most three distinct principal curvatures in \mathbb{R}^4 or in \mathbb{R}^5 are necessarily minimal (see [9, 16]) and very recently Fu (see [15]) extended Dimitric's result proving that any biharmonic hypersurface in \mathbb{R}^n with at most three distinct principal curvatures is minimal. We can now state our main result:

Theorem 1.2. *Let G be a cohomogeneity two group of isometries acting on \mathbb{R}^n ($n \geq 3$). Then any G -invariant biharmonic hypersurface in \mathbb{R}^n is minimal.*

Remark 1.3. The family of G -invariant hypersurfaces in Theorem 1.2 is ample and geometrically significant (see Table 1 below). The case $d = 1$ (i.e., the case of the classical rotational hypersurfaces) is a special instance of the above cited result of [8]. The next case, i.e., $d = 2$ and $G = SO(p) \times SO(q)$, with $p + q = n$, was proved in [20]. For these reasons, the interest of Theorem 1.2 lies on the three remaining types of G -actions, i.e., $d = 3, 4$ and 6 , for which the number of distinct principal curvatures is $4, 5$ and 7 respectively.

The paper is organized as follows: in the next section we recall several basic facts from equivariant differential geometry. We believe that this short outline could be useful also for other applications in similar contexts. In the final section we provide the details of the proof of Theorem 1.2.

2. BASIC EQUIVARIANT DIFFERENTIAL GEOMETRY FOR COHOMOGENEITY TWO G -ACTIONS ON \mathbb{R}^n

The details, together with some historical references, concerning the results of this section can be found in [1, 18, 25]. Let G be a cohomogeneity two group of isometries acting on \mathbb{R}^n . As we mentioned in the introduction, these groups are well-understood and classified since they correspond to the *isotropy representations of symmetric spaces of rank two*. For the sake of completeness, we also wish to point out the deep connection between this branch of the theory of Lie groups and the geometric properties of isoparametric functions on the Euclidean sphere (see, for example, [21, 26]).

The following linear functions $w_{(d,i)}$ will play a key role:

$$w_{(d,i)}(x, y) = x \sin(i \pi / d) - y \cos(i \pi / d) , \quad (2.1)$$

where d is an integer which can be equal to 1, 2, 3, 4 or 6, and i is another integer such that $0 \leq i \leq (d-1)$. The orbit space $Q = \mathbb{R}^n / G$ can be identified with a linear cone of angle (π/d) in \mathbb{R}^2 described by

$$Q = \{ (x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } x \sin(\pi/d) - y \cos(\pi/d) \geq 0 \} , \quad (2.2)$$

where the possible cases for d are 1, 2, 3, 4 or 6. We also point out that $Q = \mathbb{R}^2 / W$, where $W = N(H, G) / H$ is the Weil group which acts on \mathbb{R}^2 by reflections with respect to the lines defined by $w_{(d,i)}(x, y) = 0$. For this reason, the orbit space Q is also called the *Weil chamber*. The orbital distance metric g_Q on Q (i.e., the metric which makes the projection map $\Pi : \mathbb{R}^n \rightarrow Q$ a Riemannian submersion) is flat:

$$g_Q = dx^2 + dy^2 \quad (2.3)$$

and any horizontal lift of a tangent vector to Q meets any G -orbit perpendicularly. Let $\xi = (x, y)$ be an interior point of Q . We denote by $V(\xi)$ the volume of the principal orbit $\Pi^{-1}(\xi)$. The function $V(\xi)$ is called the *volume function* and contains most of the information required to carry out the computation of the second fundamental form A associated to a G -invariant hypersurface. More precisely, it turns out that $V(x, y)$ is always a homogeneous polynomial which, for each fixed type d , can be expressed (up to a multiplicative constant) in terms of the linear functions (2.1) in the following form:

$$V^2(x, y) = \prod_{i=0}^{(d-1)} [w_{(d,i)}(x, y)]^{2m_i} , \quad (2.4)$$

where the m_i 's are positive integers (the cases which can occur are listed in Table 1, which can be derived from an analogous Table given in [17, 18]).

Next, we observe that any cohomogeneity one G -invariant immersion into \mathbb{R}^n can be described by means of what we call its *profile curve* $\gamma(s) = (x(s), y(s))$ in the orbit space Q . More precisely, the G -invariant hypersurface corresponding to a profile curve γ is $\Sigma_\gamma = \Pi^{-1}(\gamma)$. For convenience, we shall always assume that

$$\dot{x}^2 + \dot{y}^2 = 1 , \quad (2.5)$$

and also, to fix orientation, that the unit normal η to the hypersurface Σ_γ projects down in Q to

$$d\Pi(\eta) = \nu = -\dot{y} \frac{\partial}{\partial x} + \dot{x} \frac{\partial}{\partial y} . \quad (2.6)$$

Now, suppose that Σ_γ is a G -invariant hypersurface into \mathbb{R}^n of type d ($d = 1, 2, 3, 4$ or 6), with associated volume function given by (2.4). Then Σ_γ possesses $(d+1)$ distinct principal curvatures given by:

$$\begin{aligned} k_i &= -\frac{1}{2} \frac{d}{d\nu} \ln [w_{(d,i)}(x, y)]^2 \\ &= -\frac{1}{2} \nu \left(\ln [w_{(d,i)}(x, y)]^2 \right) , \quad i = 0, \dots, (d-1) , \end{aligned} \quad (2.7)$$



G	Action	dim Euclidean Space	d	Multiplicities
$SO(n-1)$	$1 + \rho_{(n-1)}$	$n \geq 3$	1	$m_0 = (n-2), m_1 = 1$
$SO(p) \times SO(q)$	$\rho_p + \rho_q$	$n = (p+q) \geq 4$	2	$m_0 = (q-1), m_1 = (p-1)$
$SO(3)$	$S_{\rho_3}^2 - 1$	$n = 5$	3	$m_0 = m_1 = m_2 = 1$
$SU(3)$	Ad	$n = 8$	3	$m_0 = m_1 = m_2 = 2$
$Sp(3)$	$\Lambda^2 \nu_3 - 1$	$n = 14$	3	$m_0 = m_1 = m_2 = 4$
F_4	1 	$n = 26$	3	$m_0 = m_1 = m_2 = 8$
$SO(5)$	Ad	$n = 10$	4	$m_0 = m_1 = m_2 = m_3 = 2$
$SO(2) \times SO(m)$	$\rho_2 \otimes \rho_m$	$n = 2m \geq 6$	4	$m_0 = m_2 = (m-2), m_1 = m_3 = 1$
$S(U(2) \times U(m))$	$[\mu_2 \otimes_{\mathbb{C}} \mu_m]_{\mathbb{R}}$	$n = 4m \geq 8$	4	$m_0 = m_2 = (2m-3), m_1 = m_3 = 2$
$Sp(2) \times Sp(m)$	$\nu_2 \otimes_{\mathbb{H}} \nu_m^*$	$n = 8m \geq 16$	4	$m_0 = m_2 = (4m-5), m_1 = m_3 = 4$
$U(5)$	$[\Lambda^2 \mu_5]_{\mathbb{R}}$	$n = 20$	4	$m_0 = m_2 = 5, m_1 = m_3 = 4$
$U(1) \times Spin(10)$	$[\mu_1 \otimes_{\mathbb{C}} \Delta_1^+]_{\mathbb{R}}$	$n = 32$	4	$m_0 = m_2 = 9, m_1 = m_3 = 6$
G_2	Ad	$n = 14$	6	$m_0 = m_1 = \dots = m_5 = 2$
$SO(4)$	$1 \ 3$ 	$n = 8$	6	$m_0 = m_1 = \dots = m_5 = 1$

TABLE 1. Cohomogeneity two G -actions on \mathbb{R}^n (see [17, 18]) (**Note:** the volume function is given in (2.4), the number of distinct principal curvatures of Σ_γ is $(d+1)$).

each of them with multiplicity equal to m_i , and

$$k_d = \ddot{y} \dot{x} - \ddot{x} \dot{y} \quad (2.8)$$

with multiplicity equal to one. For future use we also observe that, using (2.1) and (2.6), (2.7) gives

$$\begin{aligned} k_i &= \frac{\dot{y} \sin(i \pi / d) + \dot{x} \cos(i \pi / d)}{w_{(d,i)}(x, y)} \\ &= \frac{w_{(d,i)}(\dot{y}, -\dot{x})}{w_{(d,i)}(x, y)}, \quad i = 0, \dots, (d-1). \end{aligned} \quad (2.9)$$

In particular, it follows that the terms f and $|A|^2$ in (1.5a) and (1.5b) are given by

$$f = (\ddot{y} \dot{x} - \ddot{x} \dot{y}) + \sum_{i=0}^{(d-1)} m_i k_i \quad (2.10)$$

and

$$|A|^2 = (\ddot{y} \dot{x} - \ddot{x} \dot{y})^2 + \sum_{i=0}^{(d-1)} m_i (k_i)^2 \quad (2.11)$$

respectively, where the explicit expressions for the k_i 's are those given in (2.9). We shall need to compute the gradient and the laplacian of f : to this purpose, we work by using on Σ_γ a local system of coordinates of type $\{u_1, \dots, u_{n-2}, s\}$, where $u = \{u_1, \dots, u_{n-2}\}$ are local coordinates of a principal orbit. In particular, we observe that, with respect to these local coordinates, the induced metric g satisfies:

$$\det g = \psi(u) V^2(x(s), y(s)), \quad (2.12)$$

where ψ is a positive function on the principal orbit and

$$g_{(n-1),k} = \delta_{(n-1),k}, \quad k = 1, \dots, (n-1). \quad (2.13)$$

Now, since f depends only on s , it follows immediately that

$$\text{grad } f = \dot{f}(s) \frac{\partial}{\partial s}. \quad (2.14)$$

Next, by using (2.12) and (2.13) in

$$\Delta f = - \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial v_i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial v_j} \right), \quad v_i = u_i, i = 1, \dots, (n-2), \quad v_{(n-1)} = s,$$

we obtain

$$\Delta f = -\ddot{f} - \frac{1}{2} \left(\frac{d}{ds} \ln V^2 \right) \dot{f}. \quad (2.15)$$

We can summarize this discussion in the following proposition which, taking into account that the Ricci tensor field of \mathbb{R}^n vanishes, follows by direct substitution of (2.10), (2.11), (2.14) and (2.15) into (1.5a) and (1.5b):

Proposition 2.1. *Let Σ_γ be a G -invariant hypersurface into \mathbb{R}^n of type d ($d = 1, 2, 3, 4$ or 6), with associated volume function given by (2.4). Then Σ_γ is a biharmonic hypersurface if and only if*

$$\ddot{f} + \frac{1}{2} \left(\frac{d}{ds} \ln V^2 \right) \dot{f} - \left[(\ddot{y} \dot{x} - \ddot{x} \dot{y})^2 + \sum_{i=0}^{(d-1)} m_i (k_i)^2 \right] f = 0 \quad (2.16)$$

and

$$\dot{f} (f + 2 (\ddot{y} \dot{x} - \ddot{x} \dot{y})) = 0 . \quad (2.17)$$

Remark 2.2. For future use, here we consider briefly the case where the profile curve $\gamma(s) = (x(s), y(s))$ in Q satisfies $y = mx$, $m \in \mathbb{R}$. Therefore, we can assume that γ is parametrized by

$$\gamma(s) = (s \cos \sigma, s \sin \sigma) , \quad (2.18)$$

where σ is a constant in the interval $(0, \pi/d)$. In this case, we have

$$f = -\frac{1}{s} \sum_{i=0}^{(d-1)} m_i \cot \left(\sigma - \frac{i\pi}{d} \right) . \quad (2.19)$$

In particular, by using (2.18) and (2.19) into (2.17), it is immediate to deduce that a curve of this type gives rise to a solution if and only if $f \equiv 0$. In other words, for this family of curves, the associated Σ_γ is biharmonic if and only if it is minimal.

Example 2.3. Equations (2.16) and (2.17) express, respectively, the vanishing of the normal and of the tangential component of the bitension field. In order to help the reader, we now compute explicitly the various principal curvatures and the biharmonicity conditions in one specific instance. More precisely, suppose that Σ_γ is a G -invariant immersion into \mathbb{R}^n of type $d = 4$, with $G = U(5)$, $n = 20$ (see Table 1). In this case we have:

$$w_{(4,0)}(x, y) = -y , \quad w_{(4,1)}(x, y) = \frac{\sqrt{2}}{2} (x - y) ,$$

$$w_{(4,2)}(x, y) = x , \quad w_{(4,3)}(x, y) = \frac{\sqrt{2}}{2} (x + y) ,$$

with $m_0 = m_2 = 5$, $m_1 = m_3 = 4$ (note that $1 + \sum_{i=0}^{(d-1)} m_i = \dim(\Sigma_\gamma) = (n - 1) = 19$). Then we have:

$$\begin{aligned} k_0 &= -\frac{1}{2} \frac{d}{d\nu} \ln [w_{(4,0)}(x, y)]^2 \\ &= -\frac{1}{w_{(4,0)}(x, y)} \left(-\dot{y} \frac{\partial}{\partial x} w_{(4,0)}(x, y) + \dot{x} \frac{\partial}{\partial y} w_{(4,0)}(x, y) \right) = -\frac{\dot{x}}{y} , \end{aligned}$$

and, similarly,

$$k_1 = \frac{\dot{x} + \dot{y}}{x - y}, \quad k_2 = \frac{\dot{y}}{x}, \quad k_3 = \frac{-\dot{x} + \dot{y}}{x + y} .$$

Moreover, up to an irrelevant multiplicative constant,

$$V^2(x, y) = \prod_{i=0}^3 [w_{(4,i)}(x, y)]^{2m_i} = (xy)^{10} (x^2 - y^2)^8 .$$

Therefore, taking into account (2.10), (2.11) and (2.15) the biharmonicity equations (2.16) and (2.17) become

$$\ddot{f} + \left(5 \frac{\dot{x}}{x} + 5 \frac{\dot{y}}{y} + 4 \frac{\dot{x} + \dot{y}}{x + y} + 4 \frac{\dot{x} - \dot{y}}{x - y} \right) \dot{f} - |A|^2 f = 0 \quad (2.20)$$

and

$$\dot{f} (f + 2(\ddot{y}\dot{x} - \ddot{x}\dot{y})) = 0 \quad (2.21)$$

respectively, where

$$f = (\ddot{y}\dot{x} - \ddot{x}\dot{y}) - 5 \frac{\dot{x}}{y} + 4 \frac{(\dot{x} + \dot{y})}{(x - y)} + 5 \frac{\dot{y}}{x} + 4 \frac{(-\dot{x} + \dot{y})}{(x + y)} \quad (2.22)$$

and

$$|A|^2 = (\ddot{y}\dot{x} - \ddot{x}\dot{y})^2 + 5 \left(\frac{\dot{x}}{y} \right)^2 + 4 \left(\frac{\dot{x} + \dot{y}}{x - y} \right)^2 + 5 \left(\frac{\dot{y}}{x} \right)^2 + 4 \left(\frac{-\dot{x} + \dot{y}}{x + y} \right)^2 . \quad (2.23)$$

3. PROOF OF THE MAIN THEOREM

Proof. Now we are in the right position to prove Theorem 1.2. The method of proof uses ideas introduced in [16] and also used in [20], where the case $d = 2$ was proved. It is enough to show that Σ_γ is a CMC immersion because direct inspection of (2.16) shows that, if we have a solution with f equal to a constant, then necessarily $f \equiv 0$ so that the immersion is minimal. So, let us assume that Σ_γ is not CMC. Then there exists a real open interval I where $\dot{f}(s) \neq 0$ for all $s \in I$, and equation (2.17) is equivalent to

$$f + 2(\ddot{y}\dot{x} - \ddot{x}\dot{y}) = 0 \quad (3.1)$$

on I . Now, in order to provide a unified proof which includes all the cases (i.e., $d = 1, 2, 3, 4, 6$), it is convenient to introduce the following function:

$$R(x, y, \dot{x}, \dot{y}) = \sum_{i=0}^{(d-1)} m_i k_i , \quad (3.2)$$

where the k_i ' are those given in (2.9). In particular, we observe from (2.10) that

$$f = R + (\ddot{y}\dot{x} - \ddot{x}\dot{y}) . \quad (3.3)$$

Next, from equation (3.1) we deduce that

$$\ddot{y}\dot{x} - \ddot{x}\dot{y} = -\frac{R}{3} \quad (3.4)$$

on I . Therefore, multiplying by \dot{x} both sides of (3.4) and using (2.5), we easily obtain

$$\ddot{y} = -\frac{R}{3} \dot{x} . \quad (3.5)$$

In the same way, multiplying by \dot{y} , we also have

$$\ddot{x} = \frac{R}{3} \dot{y} . \quad (3.6)$$

We also observe that, using (3.5) and (3.6), we can rewrite the relationship between f and R as follows:

$$f = \frac{2}{3} R . \quad (3.7)$$

Now, using (3.4)–(3.7), which come from the tangent component (2.17), we claim that the normal component (2.16) can be written in the following equivalent form:

$$A_0(x, y) \dot{x}^3 + A_1(x, y) \dot{x}^2 \dot{y} + A_2(x, y) \dot{x} \dot{y}^2 + A_3(x, y) \dot{y}^3 = 0 , \quad (3.8)$$

where the $A_j(x, y)$'s ($j = 0, \dots, 3$) are homogeneous polynomials of degree $3(d-1)$. In order to verify this claim, one must simply carry out an explicit computation. More precisely, first, by using (3.4) and (3.7), we rewrite equation (2.16) in terms of R as follows:

$$\ddot{R} + \frac{1}{2} \left(\frac{d}{ds} \ln V^2 \right) \dot{R} - \left[(1/9) R^2 + \sum_{i=0}^{(d-1)} m_i (k_i)^2 \right] R = 0 . \quad (3.9)$$

Next, we define the following homogeneous polynomial of degree d in the variables x, y :

$$Q_d = \prod_{i=0}^{(d-1)} w_{(d,i)}(x, y) \quad (3.10)$$

and also observe that

$$\frac{d}{ds} w_{(d,i)}(x, y) = w_{(d,i)}(\dot{x}, \dot{y}) . \quad (3.11)$$

Now, we analyse the single terms which appear in (3.9). Using (3.11) and (2.4) a direct computation gives

$$\frac{1}{2} \left(\frac{d}{ds} \ln V^2 \right) = \sum_{i=0}^{(d-1)} m_i \frac{w_{(d,i)}(\dot{x}, \dot{y})}{w_{(d,i)}(x, y)} = \frac{T_{1,(d-1)} \dot{x} + T_{2,(d-1)} \dot{y}}{Q_d} , \quad (3.12)$$

where $T_{1,(d-1)}$ and $T_{2,(d-1)}$ are the homogeneous polynomials of degree $(d-1)$ in the variables x, y given by

$$T_{1,(d-1)} = \sum_{i=0}^{(d-1)} m_i \frac{Q_d}{w_{(d,i)}(x, y)} \sin(i\pi/d) ,$$

$$T_{2,(d-1)} = - \sum_{i=0}^{(d-1)} m_i \frac{Q_d}{w_{(d,i)}(x, y)} \cos(i\pi/d) .$$

Next, using (2.9) in (3.2) we obtain

$$R = \frac{-T_{2,(d-1)} \dot{x} + T_{1,(d-1)} \dot{y}}{Q_d} . \quad (3.13)$$

Taking the square of (3.13) and using again (2.9), we also find

$$(1/9) R^2 + \sum_{i=0}^{(d-1)} m_i (k_i)^2 = \frac{T_{3,(2d-2)} \dot{x}^2 + T_{4,(2d-2)} \dot{x} \dot{y} + T_{5,(2d-2)} \dot{y}^2}{Q_d^2}, \quad (3.14)$$

where $T_{3,(2d-2)}$, $T_{4,(2d-2)}$ and $T_{5,(2d-2)}$ are the homogeneous polynomials of degree $(2d-2)$ in the variables x, y given by

$$\begin{aligned} T_{3,(2d-2)} &= \frac{1}{9} T_{2,(d-1)}^2 + \sum_{i=0}^{(d-1)} m_i \frac{Q_d^2}{[w_{(d,i)}(x, y)]^2} \cos^2(i\pi/d) \\ T_{4,(2d-2)} &= -\frac{2}{9} T_{1,(d-1)} T_{2,(d-1)} + 2 \sum_{i=0}^{(d-1)} m_i \frac{Q_d^2}{[w_{(d,i)}(x, y)]^2} \sin(i\pi/d) \cos(i\pi/d) \\ T_{5,(2d-2)} &= \frac{1}{9} T_{1,(d-1)}^2 + \sum_{i=0}^{(d-1)} m_i \frac{Q_d^2}{[w_{(d,i)}(x, y)]^2} \sin^2(i\pi/d). \end{aligned}$$

Now, taking the first and the second derivatives of (3.13) and using (3.5) and (3.6), a direct computation shows that

$$\dot{R} = \frac{T_{6,(2d-2)} \dot{x}^2 + T_{7,(2d-2)} \dot{x} \dot{y} + T_{8,(2d-2)} \dot{y}^2}{Q_d^2}, \quad (3.15)$$

$$\ddot{R} = \frac{T_{9,(3d-3)} \dot{x}^3 + T_{10,(3d-3)} \dot{x}^2 \dot{y} + T_{11,(3d-3)} \dot{x} \dot{y}^2 + T_{12,(3d-3)} \dot{y}^3}{Q_d^3}, \quad (3.16)$$

where $T_{6,(2d-2)}, T_{7,(2d-2)}, T_{8,(2d-2)}$ are homogeneous polynomials of degree $(2d-2)$ in the variables x, y , while $T_{9,(3d-3)}, T_{10,(3d-3)}, T_{11,(3d-3)}, T_{12,(3d-3)}$ are homogeneous polynomials of degree $(3d-3)$ in the variables x, y .

Finally, using (3.12)–(3.16) into (3.9), one obtains (up to a denominator Q_d^3) that the claimed equation (3.8) holds (note that the explicit expressions for the $A_j(x, y)$'s play no active role in the sequel, so we omit further details on this point).

Next, for a fixed $s_0 \in I$, we put $x_0 = x(s_0)$. Since $\dot{x}^2 + \dot{y}^2 = 1$, we can assume that $\dot{x}(s_0) \neq 0$ and we can express y as a function of x , $y = y(x)$, with $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, and write

$$\dot{y} = \frac{dy}{dx} \dot{x}. \quad (3.17)$$

From $\dot{x}^2 + \dot{y}^2 = 1$ we also obtain

$$\dot{x}^2 = \frac{1}{1 + \left(\frac{dy}{dx}\right)^2}. \quad (3.18)$$

For future use, combining (3.17) and (3.18), we also observe that

$$\dot{x} \dot{y} = \frac{1}{1 + \left(\frac{dy}{dx}\right)^2} \frac{dy}{dx}. \quad (3.19)$$

Multiplying (3.8) by \dot{x} and substituting (3.18) and (3.19) we find that, up to a multiplicative factor $1/\left(1 + (dy/dx)^2\right)^2$, (3.8) becomes equivalent to

$$A_3(x, y) \left(\frac{dy}{dx}\right)^3 + A_2(x, y) \left(\frac{dy}{dx}\right)^2 + A_1(x, y) \left(\frac{dy}{dx}\right) + A_0(x, y) = 0 . \quad (3.20)$$

Next, deriving (3.17) with respect to s , a straightforward computation leads us to

$$\ddot{y} = \frac{1}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^2} \frac{d^2 y}{dx^2} . \quad (3.21)$$

Next, we use (3.5) into (3.21): a computation, which takes into account (3.13), (3.18) and (3.19), shows that the expression for the second derivative of y with respect to x is given by:

$$\begin{aligned} \frac{d^2 y}{dx^2} &= -\frac{1}{3} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^2 R \dot{x} \\ &= -\frac{1}{3} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^2 \left(\frac{-T_{2,(d-1)}\dot{x} + T_{1,(d-1)}\dot{y}}{Q_d}\right) \dot{x} \\ &= \frac{1}{3} \left(1 + \left(\frac{dy}{dx}\right)^2\right) \left(\frac{T_{2,(d-1)} - T_{1,(d-1)}(dy/dx)}{Q_d}\right) . \end{aligned} \quad (3.22)$$

Next, taking the derivative of (3.20) with respect to x , that is d/dx , and using (3.22), we obtain, up to $1/Q_d$, the following equation:

$$\begin{aligned} &C_5(x, y) \left(\frac{dy}{dx}\right)^5 + C_4(x, y) \left(\frac{dy}{dx}\right)^4 \\ &+ C_3(x, y) \left(\frac{dy}{dx}\right)^3 + C_2(x, y) \left(\frac{dy}{dx}\right)^2 + C_1(x, y) \left(\frac{dy}{dx}\right) + C_0(x, y) = 0 , \end{aligned} \quad (3.23)$$

where the $C_j(x, y)$'s ($j = 0, \dots, 5$) are homogeneous polynomials of degree $4(d-1)$ which are related to the $A_j(x, y)$'s as follows:

$$\begin{cases} C_0 = Q_d \cdot \partial A_0 / \partial x + (1/3) A_1 \cdot T_{2,(d-1)} \\ C_1 = Q_d \cdot (\partial A_1 / \partial x + \partial A_0 / \partial y) + (2/3) A_2 \cdot T_{2,(d-1)} - (1/3) A_1 \cdot T_{1,(d-1)} \\ C_2 = Q_d \cdot (\partial A_2 / \partial x + \partial A_1 / \partial y) + A_3 \cdot T_{2,(d-1)} - (2/3) A_2 \cdot T_{1,(d-1)} + (1/3) A_1 \cdot T_{2,(d-1)} \\ C_3 = Q_d \cdot (\partial A_3 / \partial x + \partial A_2 / \partial y) - A_3 \cdot T_{1,(d-1)} + (2/3) A_2 \cdot T_{2,(d-1)} - (1/3) A_1 \cdot T_{1,(d-1)} \\ C_4 = Q_d \cdot \partial A_3 / \partial y + A_3 \cdot T_{2,(d-1)} - (2/3) A_2 \cdot T_{1,(d-1)} \\ C_5 = -A_3 \cdot T_{1,(d-1)} , \end{cases} \quad (3.24)$$

where in this computation we have used

$$\frac{dA_i}{dx} = \frac{\partial A_i}{\partial x} + \frac{\partial A_i}{\partial y} \frac{dy}{dx} , \quad i = 0, \dots, 3 .$$

Now we are in the right position to end the proof: for any arbitrarily fixed $x_1 \in (x_0 - \varepsilon, x_0 + \varepsilon)$, setting $y_1 = y(x_1)$, (3.20) and (3.23) can be thought of as two polynomial equations in dy/dx , with coefficients given, respectively, by $A_j(x_1, y_1)$, $j = 0, \dots, 3$ and $C_j(x_1, y_1)$, $j = 0, \dots, 5$, which have the common solution $(dy/dx)(x_1)$. Using standard arguments of algebraic geometry ([16]), this implies that the resultant of the two polynomials is zero for any $x_1 \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Now, since the coefficients $A_j(x, y)$ and $C_j(x, y)$ are *homogeneous polynomials* (of degree $3(d-1)$ and $4(d-1)$ respectively), it turns out that the resultant is itself a *homogeneous polynomial* of degree $3(d-1) \cdot 5 + 4(d-1) \cdot 3 = 27(d-1)$. Then, we can divide the resultant by $x^{27(d-1)}$ and putting $z = y/x$ we obtain a polynomial equation in z with constant coefficients. Since z is continuous it must be constant, that is $y = mx$, $m \in \mathbb{R}$. But any such solution is biharmonic if and only if it is minimal (see Remark 2.2): a contradiction with the hypothesis that Σ_γ is not CMC. \square

Remark 3.1. We think that it is important to stress the fact the method of proof works because the G -invariance is sufficient to guarantee that the biharmonicity conditions (2.16) and (2.17), when expressed in the form (3.20) and (3.23), have coefficients given by *homogeneous* polynomials. In the direction of proving the Chen conjecture, we also point out that the incompatibility between (2.16) and (2.17) appear to be of a local nature.

Example 3.2. For the sake of completeness, we report here the explicit expressions of the relevant homogeneous polynomials which appear in the Example 2.3. In this case we have:

$$A_1(x, y) = A_2(y, x) = 8(-25x^2y^7 + 540x^4y^5 - 118x^6y^3 + 65x^8y - 10y^9),$$

$$A_0(x, y) = A_3(y, x) = 775x^7y^2 - 363x^5y^4 + 1237x^3y^6 + 130xy^8 - 275x^9.$$

REFERENCES

- [1] A. Back, M.P. do Carmo, W.-Y. Hsiang. On some fundamental equations of equivariant Riemannian geometry. *Tamkang J. Math.* 40 (2009), 343–376.
- [2] A. Balmuş, S. Montaldo, C. Oniciuc. Biharmonic PNMC submanifolds in spheres. *Ark. Mat.* 51 (2013), 197–221.
- [3] E. Bombieri, E. De Giorgi, E. Giusti. Minimal cones and the Bernstein problem. *Invent. Math.* 7 (1969), 243–268.
- [4] B.-Y. Chen. *Total mean curvature and submanifolds of finite type*, second edition. Serier in Pure Mathematics, Vol. 27 (2015).
- [5] B.-Y. Chen. Recent developments of biharmonic conjecture and modified biharmonic conjectures. arXiv:1307.0245, to appear in Proceedings of PADGE-2012.
- [6] B.-Y. Chen. Some open problems and conjectures on submanifolds of finite type. *Soochow J. Math.* 17 (1991), 169–188.
- [7] B.-Y. Chen. *Total Mean Curvature and Submanifolds of Finite Type*. Series in Pure Mathematics 1. World Scientific Publishing Co., Singapore, 1984.
- [8] I. Dimitric. Submanifolds of E^m with harmonic mean curvature vector. *Bull. Inst. Math. Acad. Sinica* 20 (1992), 53–65.
- [9] F. Defever. Hypersurfaces of E^4 with harmonic mean curvature vector. *Math. Nachr.* 196 (1998), 61–69.

- [10] J. Eells, L. Lemaire. *Selected topics in harmonic maps*. CBMS Regional Conference Series in Mathematics, 50. American Mathematical Society, Providence, RI, 1983.
- [11] J. Eells, J.H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* 86 (1964), 109–160.
- [12] J. Eells, A. Ratto. Harmonic maps and minimal immersions with symmetries. *Ann. Math. Studies* 133, Princeton University Press (1993), 1–228.
- [13] G.Y. Jiang. 2-harmonic maps and their first and second variation formulas. *Chinese Ann. Math. Ser. A* 7, 7 (1986), 130–144.
- [14] Y. Fu. Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space. *J. Geom. Phys.* 75 (2014), 113–119.
- [15] Y. Fu. Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space. *Tohoku Math. J.* in press.
- [16] Th. Hasanis, Th. Vlachos. Hypersurfaces in E^4 with harmonic curvature vector field. *Math. Nachr.* 172 (1995), 145–169.
- [17] W.-Y. Hsiang. Generalized rotational hypersurfaces of constant mean curvature in the Euclidean space I. *J. Diff. Geom.* 17 (1982), 337–356.
- [18] W.-Y. Hsiang, B. Lawson. Minimal submanifolds of low cohomogeneity. *J. Diff. Geom.* 5 (1971), 1–38.
- [19] E. Loubeau, S. Montaldo. Biminimal immersions. *Proc. Edinb. Math. Soc.* 51 (2008), 421–437.
- [20] S. Montaldo, C. Oniciuc, A. Ratto. Proper biconservative immersions into the Euclidean space. *Ann. Mat. Pura Appl.*, in press.
- [21] H.-F. Münzner. Isoparametrische Hyperflächen in Sphären. *Math. Ann.* 251 (1980), 57–71.
- [22] S. Myers, N. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math.* 40 (1939), 400–416.
- [23] Y.-L. Ou. Biharmonic hypersurfaces in Riemannian manifolds. *Pacific J. Math.* 248 (2010), 217–232.
- [24] C. Oniciuc. Biharmonic maps between Riemannian manifolds. *An. Stiint. Univ. Al.I. Cuza Iasi Mat (N.S.)* 48 (2002), 237–248.
- [25] R.H.L. Pedrosa. Riemannian transformation groups. In *Lectures on isometric actions*, XV Esc. Geom. Diferencial, pages 5–75. Inst. Mat. Pura Apl. (IMPA), Rio de Janeiro, 2008.
- [26] G. Thorbergsson. A survey on isoparametric hypersurfaces and their generalizations. In *Handbook of Differential Geometry*, Vol. I, Editors F. Dillen and L. Verstraelen, Elsevier Science, Amsterdam, 2000, 963–995.

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